

Pentapods with Mobility 2

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Abstract

In this paper we give a full classification of all pentapods with mobility 2, where neither all platform anchor points nor all base anchor points are located on a line. Therefore this paper solves the famous Borel-Bricard problem for 2-dimensional motions beside the excluded case of five collinear points with spherical trajectories. But even for this special case we present three new types as a side-result. Based on our study of pentapods, we also give a complete list of all non-architecturally singular hexapods with 2-dimensional self-motions.

Keywords: Pentapod, Self-Motion, Borel-Bricard Problem, Bond Theory, Hexapod

1. Introduction

The geometry of a pentapod (see Fig. 1) is given by the five base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed system and by the five platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$ with respect to the moving system (for $i = 1, \dots, 5$). Each pair (M_i, m_i) of corresponding anchor points is connected by a SPS-leg, where only the prismatic joint (P) is active and the spherical joints (S) are passive.

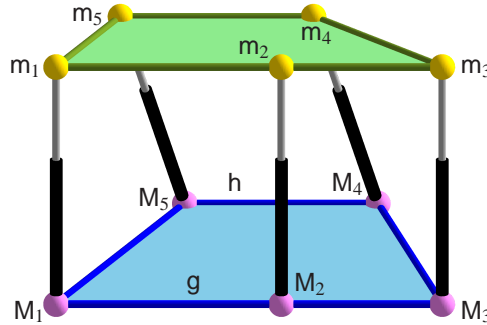


Figure 1: Sketch of a pentapod with planar platform and planar base, which is referred as planar pentapod. Moreover this planar pentapod has a 2-dimensional self-motion due to its special geometric design (cf. item 2 of Theorem 3).

If the geometry of the manipulator is given, as well as the lengths of the five pairwise distinct legs, the pentapod has generically mobility 1 according to the formula of Grübler. The corresponding motion is called a 1-dimensional self-motion of the pentapod. But, under particular conditions, the manipulator can gain additional mobility. These pentapods represent interesting solutions to the still unsolved problem posed by the French Academy of Science for the *Prix Vaillant* of the year 1904, which is also known as Borel-Bricard problem (cf. [1, 2, 3]) and reads as follows: *”Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.”*

1.1. Related results

The classification of pentapods with mobility 2 is based on the following theorem proven in [4].

Theorem 1. *If the mobility of a pentapod is 2 or higher, then one of the following conditions holds¹:*

- (a) *The platform and the base are similar. This is a so-called equiform pentapod.*
- (b) *The platform and the base are planar and affine equivalent. This is a so-called planar affine pentapod.*
- (c) *There exists $p \leq 5$ such that m_1, \dots, m_p are collinear and M_{p+1}, \dots, M_5 are equal; i.e. $M_{p+1} = \dots = M_5$.*
- (d) *M_1, M_2, M_3 are located on the line g which is parallel to the line h spanned by M_4 and M_5 . Moreover m_1, m_2, m_3 are located on the line g' which is parallel to the line h' spanned by m_4 and m_5 .*

We can focus on pentapods with 2-dimensional self-motions, as those with higher-dimensional ones follow from Theorem 7 of [5]. They all belong to item (c) of Theorem 1 and read as follows (for more details please see [5]):

Corollary 1. A pentapod has an n -dimensional self-motion with $n > 2$ in one of the following three cases (under consideration of footnote 1):

- (i) $m_1 = m_2 = m_3$ and $M_4 = M_5$.
- (ii) $m_1 = m_2 = m_3 = m_4$.
- (iii) All base points are collinear, all platform points are collinear and corresponding anchor points are related by a regular projectivity: $m_i \mapsto M_i$ for $i = 1, \dots, 5$.

Moreover, we can restrict to the case $p \leq 4$ in item (c) of Theorem 1, as we only study pentapods where neither all platform anchor points nor all base anchor points are collinear. The reason for excluding $p = 5$ is that it requires a special treatment, as a 1-dimensional self-motion of the carrier line g of m_1, \dots, m_5 already implies mobility 2 of the moving space, as in each pose of g a 1-dimensional rotation about it is possible. Therefore the self-motions of pentapods with linear platform are studied separately in [6].

In the following we list all pentapods² with mobility 2, which are known in the literature until now, with respect to the three items of Theorem 1:

Examples ad (a), (b) and (d). The platform and the base are congruent (= so-called congruent pentapod) and all legs have the same length. This pentapod, which belongs to case (a), can perform a 2-dimensional translational self-motion.

If the platform and base have the additional property to be planar, then we get an example which also belongs to item (b). In the special case, where the anchor points are located on two parallel lines, the pentapod possesses an additional 2-dimensional self-motion, which is neither pure translational nor pure spherical (cf. Example 2 of [5] after the removal of one leg). This example also belongs to item (d) of Theorem 1.

Examples ad (c). The remaining known examples belong to this class and read as follows (under consideration of footnote 1):

- Architecturally singular³ pentapods have at least a 2-dimensional self-motion in each pose over \mathbb{C} as they are redundant. For the listing of these manipulators see Theorem 3 of [8] under consideration of [9]. As all these cases are known, they are not of further interest. In this context it should also be noted that the designs (ii) and (iii) of Corollary 1 are also architecturally singular ones.
- $m_1 = m_2$ and $M_3 = M_4$: If the platform is placed in a way that $m_1 = m_2$ coincides with $M_3 = M_4$, then there exists a 2-dimensional spherical self-motion (cf. [5]).
- For $p = 2$ the condition given in item (c) of Theorem 1 is already sufficient, as there exists a 2-dimensional spherical self-motion if the platform is placed in a way that m_1 or m_2 coincide with $M_3 = M_4 = M_5$ (cf. [5]). Note that we get architecturally singular pentapods for $p < 2$ (cf. item (ii) of Corollary 1).

¹After a possible necessary renumbering of anchor points and exchange of the platform and the base.

²In the remainder of the article the word pentapod denotes a 5-legged manipulator with non-collinear platform points and non-collinear base points; exceptions are noted explicitly.

³A pentapod (hexapod) is called architecturally singular if in any pose of the platform the rank of its Jacobian matrix J is less than five (six), which is equivalent with the statement that the carrier lines of the five (six) legs belong to a linear congruence of lines (linear line complex). This equivalence is easy to see, as J is composed of the Plücker coordinates of these five (six) lines (cf. [7]).

1.2. Results and outline of the article

In this paper we study the three cases given in Theorem 1 in more detail. The obtained results are summarized in the following three theorems:

Theorem 2. *A non-planar equiform pentapod can only have a 2-dimensional self-motion in the special case of congruent platform and base: The 2-dimensional self-motion is the trivial 2-dimensional translation.*

Theorem 3. *A planar affine pentapod has mobility 2 in one of the following two cases (under consideration of footnote 1), if the pentapod is assumed to be not architecturally singular (\Leftrightarrow no 4 points are collinear):*

1. *The affinity is a congruence: In this case the 2-dimensional self-motion is pure translational (planar case of Theorem 2).*
2. *M_1, M_2, M_3 are located on the line \mathbf{g} which is parallel to the line \mathbf{h} spanned by M_4 and M_5 . Moreover the restriction of the affinity to the lines is a congruence transformation. The 2-dimensional self-motion is a Schönflies motion, where the axis of rotation is parallel to \mathbf{g}, \mathbf{h} and the corresponding two lines in the platform (see Fig. 1).*

Theorem 4. *A non-architecturally singular pentapod with mobility 2, which is not listed in Theorems 2 and 3, has to be one of the following designs (under consideration of footnote 1):*

1. *$p = 2$: The given condition $M_3 = M_4 = M_5$ is already sufficient for mobility 2. The 2-dimensional spherical self-motion is obtained if $M_3 = M_4 = M_5$ coincides with \mathbf{m}_1 or \mathbf{m}_2 .*
2. *$p = 3$; i.e. $M_4 = M_5$ and $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are collinear: In the following two cases we get mobility 2:*
 - (a) *$\mathbf{m}_2 = \mathbf{m}_3$: If $M_4 = M_5$ coincides with $\mathbf{m}_2 = \mathbf{m}_3$ we obtain the 2-dimensional spherical self-motion.*
 - (b) *$M_2, M_3, M_4 = M_5$ collinear: If $M_4 = M_5$ coincides with \mathbf{m}_1 we obtain the 2-dimensional spherical self-motion.*
3. *$p = 4$; i.e. $\mathbf{m}_1, \dots, \mathbf{m}_4$ are collinear: For M_2, M_3, M_4, M_5 collinear we get mobility 2. The 2-dimensional spherical self-motion is obtained if \mathbf{m}_1 coincide with M_5 .*

This complete classification of pentapods with mobility 2 reveals new cases, which can easily be seen by comparing the listed designs with the known results. It should be mentioned that item 2 of Theorem 3 is the generalization of the already mentioned Example 2 of [5]. Note that this example is the only one, which possesses both 2-dimensional self-motions given in Theorem 3.

Remark 1. As item 2 of Theorem 3 is known now, its existence can also easily be argued from the following property implied by the Schönflies motion group (cf. [10]): Every leg can be translated in direction of the rotation axis of the Schönflies self-motion without changing this motion. Therefore the five legs only imply two constraints to the 4-dimensional Schönflies motion group. From this point of view this solution is also trivial. \diamond

The proof of Theorem 2 and 3, which is given in Section 3 and 4, respectively, is based on the theory of bonds presented in Section 2. The proof of Theorem 4 is given in Section 5 and consists of two parts: In the first part (cf. Sections 5.1), item (c) of Theorem 1 is studied by pure geometric-kinematic considerations/arguments, where also new results on self-motions of pentapods with linear platform are obtained as a side-result. In the second part (cf. Sections 5.2 and Appendix), item (d) of Theorem 1 is again discussed by means of bond theory.

Based on Theorems 2, 3 and 4 we give a complete list of all non-architecturally singular hexapods (cf. footnote 3) with mobility 2 in Section 6.

2. Bond Theory

In Section 2.1, we give a short introduction to the theory of bonds for pentapods presented in [11], which was motivated by the bond theory of overconstrained closed linkages with revolute joints given in [12] (see also [13]). We start with the direct kinematic problem of pentapods and proceed with the definition of bonds. Based on these basics, we do some preparatory work in Section 2.2 by giving a classification of 2-dimensional self-motions, which is induced by the bond theory in a natural way.

2.1. Definition of bonds

Due to the result of Husty [14], it is advantageous to work with Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ for solving the forward kinematics. Note that the first four homogeneous coordinates $(e_0 : e_1 : e_2 : e_3)$ are the so-called Euler parameters. Now, all real points of the Study parameter space P^7 (7-dimensional projective space), which are located on the so-called Study quadric $\Psi : \sum_{i=0}^3 e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the 3-dimensional subspace E of Ψ given by $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $N \neq 0$ with $N = e_0^2 + e_1^2 + e_2^2 + e_3^2$. The translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and the rotation matrix \mathbf{R} of the corresponding Euclidean displacement $\mathbf{x} \mapsto \mathbf{R}\mathbf{x} + \mathbf{t}$ are given by:

$$t_1 = 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \quad t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3), \quad t_3 = 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1),$$

and

$$\mathbf{R} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}, \quad (1)$$

if the normalizing condition $N = 1$ is fulfilled. All points of the complex extension of P^7 , which cannot fulfill this normalizing condition, are located on the so-called exceptional cone $N = 0$ with vertex E .

By using the Study parametrization of Euclidean displacements, the condition that the point \mathbf{m}_i is located on a sphere centered in \mathbf{M}_i with radius R_i is a quadratic homogeneous equation according to Husty [14]. This so-called sphere condition Λ_i has the following form:

$$\begin{aligned} \Lambda_i : & (a_i^2 + b_i^2 + c_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)N - 2(a_i A_i + b_i B_i + c_i C_i)e_0^2 - 2(a_i A_i - b_i B_i - c_i C_i)e_1^2 \\ & + 2(a_i A_i - b_i B_i + c_i C_i)e_2^2 + 2(a_i A_i + b_i B_i - c_i C_i)e_3^2 + 4(c_i B_i - b_i C_i)e_0 e_1 - 4(c_i A_i - a_i C_i)e_0 e_2 \\ & + 4(b_i A_i - a_i B_i)e_0 e_3 - 4(b_i A_i + a_i B_i)e_1 e_2 - 4(c_i A_i + a_i C_i)e_1 e_3 - 4(c_i B_i + b_i C_i)e_2 e_3 \\ & + 4(a_i - A_i)(e_0 f_1 - e_1 f_0) + 4(b_i - B_i)(e_0 f_2 - e_2 f_0) + 4(c_i - C_i)(e_0 f_3 - e_3 f_0) + 4(a_i + A_i)(e_3 f_2 - e_2 f_3) \\ & + 4(b_i + B_i)(e_1 f_3 - e_3 f_1) + 4(c_i + C_i)(e_2 f_1 - e_1 f_2) + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0. \end{aligned} \quad (2)$$

Now the solution for the direct kinematics over \mathbb{C} of a pentapod can be written as the algebraic variety V of the ideal spanned by $\Psi, \Lambda_1, \dots, \Lambda_5, N = 1$. In general V consists of a 1-dimensional set of points.

We consider the algebraic motion of the pentapod, which is defined as the set of points on the Study quadric determined by the constraints; i.e. the common points of the six quadrics $\Psi, \Lambda_1, \dots, \Lambda_5$. Now the points of the algebraic motion with $N \neq 0$ equal the kinematic image of the algebraic variety V . But we can also consider the points of the algebraic motion, which belong to the exceptional cone $N = 0$. An exact mathematical definition of these so-called bonds can be given as follows (cf. Definition 1 of [11]):

Definition 1. For a pentapod the set \mathcal{B} of bonds is defined as:

$$\mathcal{B} := \text{ZarClo}(V^*) \cap \{(e_0 : \dots : f_3) \in P^7 \mid \Psi, \Lambda_1, \dots, \Lambda_5, N = 0\},$$

where V^* denotes the variety V after the removal of all components, which correspond to pure translational motions. Moreover $\text{ZarClo}(V^*)$ is the Zariski closure of V^* , i.e. the zero locus of all algebraic equations that also vanish on V^* .

The restriction to non-translational motions is caused by the following approach used for the computation of bonds: In a first step we project the algebraic motion of the pentapod into the Euler parameter space P^3 by the elimination of f_0, \dots, f_3 . This projection is denoted by π_f . In a second step we determine those points of the projected point set $\pi_f(V)$, which are located on the quadric $N = 0$; i.e.

$$\mathcal{B}_f := \text{ZarClo}(\pi_f(V)) \cap \{(e_0 : \dots : e_3) \in P^3 \mid N = 0\}. \quad (3)$$

Note that this set of projected bonds, which is denoted by \mathcal{B}_f , cannot be empty for an non-translational self-motion.

Clearly, the kernel of this projection π_f equals the group of translational motions. As a consequence a component of V , which corresponds to a pure translational motion, is projected to a single point \mathbf{O} (with $N \neq 0$) of the Euler parameter space P^3 by the elimination of f_0, \dots, f_3 . Therefore the intersection of \mathbf{O} and $N = 0$ equals \emptyset , which reasons the exclusion of pure translational motions within this approach.

Moreover it is important to note that the set of bonds only depends on the geometry of the pentapod, and not on the leg lengths (cf. Theorem 1 of [11]).

Remark 2. A more sophisticated bond theory for pentapods and hexapods is based on a special compactification of $SE(3)$, where the sphere condition Λ_i is only linear in 17 motion-parameters. This approach, which was presented in [15], has many theoretical advantages compared to the method described above, but it is not suited for direct computations due to the large number of motion-parameters. In contrast the approach of the paper at hand was already successfully used for direct computations in [5, 11, 16, 17]. \diamond

Due to Theorem 2⁴ of [11] a pentapod possesses a pure translational self-motion, if and only if the platform can be rotated about the center $\mathbf{m}_1 = \mathbf{M}_1$ into a pose, where the vectors $\overrightarrow{\mathbf{M}_i\mathbf{m}_i}$ for $i = 2, \dots, 5$ fulfill the condition $rk(\overrightarrow{\mathbf{M}_2\mathbf{m}_2}, \dots, \overrightarrow{\mathbf{M}_5\mathbf{m}_5}) \leq 1$. Moreover all 1-dimensional self-motions are circular translations, which can easily be seen by considering a normal projection of the manipulator in direction of the parallel vectors $\overrightarrow{\mathbf{M}_i\mathbf{m}_i}$ for $i = 2, \dots, 5$. If all these five vectors are zero-vectors, which corresponds with the case that the platform and the base are congruent, we get the already mentioned 2-dimensional translational self-motion of a pentapod. This finishes already the discussion of pentapods with pure translatoric self-motions.

2.2. Classification of 2-dimensional self-motions

We assume that a given pentapod has a 2-dimensional self-motion \mathcal{S} . As \mathcal{S} corresponds with a 2-dimensional solution of the direct kinematics problem, the corresponding algebraic motion is also 2-dimensional. Due to the fact that this algebraic motion is the kinematic image of a self-motion, it cannot be located within the exceptional cone $N = 0$. Therefore the bond-set \mathcal{B} of this self-motion has to be an algebraic variety of dimension 1; i.e. a *bonding curve*.

Now we want to classify \mathcal{S} with respect to the dimension β of \mathcal{B}_f . This classification was already used successfully for the determination of all hexapods with $(n > 2)$ -dimensional self-motions (cf. [5]).

As we have a bonding curve in the Study parameter space P^7 , β can take the values $-1, 0, 1$, where the case $\beta = 1$ is the general one. In order that $\beta = i$ holds for $i = -1, 0$, there has to exist a $(1 - i)$ -dimensional translational sub-self-motion, which is contained in \mathcal{S} , in each pose of \mathcal{S} . For $i = -1$ this already implies that \mathcal{S} is a 2-dimensional translation; i.e. that platform and base are congruent. As $\beta = -1$ is already known, we only have to check the cases $\beta = 1$ and $\beta = 0$.

Remark 3. Note that the self-motions given in Theorem 2 and item 1 of Theorem 3 are of type $\beta = -1$. We will see within the proof of Theorem 3 that the self-motion of item 2 is of type $\beta = 0$ (see also Example 2 of [5]). All designs of Theorem 4 are of type $\beta = 1$ as the 2-dimensional self-motions are spherical. \diamond

3. Proving Theorem 2

Due to the above given considerations we only have to discuss the cases $\beta = 0$ and $\beta = 1$.

3.1. $\beta = 0$

As there exists a 1-dimensional translational sub-self-motion in each pose of the 2-dimensional self-motion, we can apply Theorem 2 of [11] (cf. footnote 4). As a consequence the platform and the base of the pentapod have to be related by a congruence transformation, which can be:

⁴This theorem is originally stated for hexapods but it also holds for pentapods, as its proof is also valid for 5-legged manipulators.

3.1.1. Orientation preserving (\Rightarrow congruent pentapod)

Due to Lemma 1 of [16] (cf. footnote 4) non-planar congruent pentapods cannot have a 1-dimensional translational self-motion. Therefore the case $\beta = 0$ cannot exist.

3.1.2. Orientation reversing (\Rightarrow reflection-congruent pentapod)

In this case we can apply Theorem 2 of [17] (cf. footnote 4). Therefore non-planar reflection-congruent pentapods possess a 2-parametric set of orientations with 1-dimensional translational self-motions. In the following we show by means of computation that $\mathcal{B}_f = \emptyset$ holds for these manipulators, which implies the non-existence of 2-dimensional self-motion of type $\beta = 0$:

Without loss of generality (w.l.o.g.) we can assume that the first four anchor points span a tetrahedron. Moreover we can choose special coordinate systems in the platform and the base in a way that we get:

$$\begin{aligned} \mathbf{m}_1 = \mathbf{M}_1 &= (0, 0, 0)^T, & \mathbf{M}_j &= (a_j, b_j, c_j)^T, \\ \mathbf{m}_2 = \mathbf{M}_2 &= (a_2, 0, 0)^T, & \mathbf{m}_j &= (a_j, b_j, -c_j)^T, \\ \mathbf{m}_3 = \mathbf{M}_3 &= (a_3, b_3, 0)^T, \end{aligned}$$

with $b_3c_4 \neq 0$ and $j = 4, 5$. In addition we can eliminate the factor of similarity by setting $a_2 = 1$. For this setup, it can easily be seen that the 2-parametric set of platform orientations, which cause 1-dimensional translational sub-self-motions, is determined by $e_3 = 0$. We have to distinguish the following cases:

1. $e_2 \neq 0$: Under this assumption we can solve $\Psi, \Delta_{2,1}$ for f_2, f_3 w.l.o.g., where $\Delta_{i,j} := \Lambda_i - \Lambda_j$ is only linear in f_0, \dots, f_3 . We plug the obtained solutions into $\Delta_{3,1}, \Delta_{4,1}$ and $\Delta_{5,1}$. The numerators of the resulting expressions are denoted by G_3, G_4 and G_5 , respectively, which are homogeneous cubic polynomials in e_0, e_1, e_2 . Note that G_3, G_4 and G_5 do not depend on f_0 and f_1 . These two Study parameters only appear in Λ_1 , but this equation is not of interest for the further computation of bonds.

We eliminate e_0 from G_i by calculating the resultant H_i of G_i and N with respect to e_0 for $i = 3, 4$. Now H_3 can only vanish without contradiction (w.c.) for either $e_1 = \frac{a_3}{b_3}e_2$ or $e_1 = \frac{a_3-1}{b_3}e_2$. In both cases H_4 has to be fulfilled identically. The resulting condition can in both cases be solved for a_4 w.l.o.g., but it can easily be seen that none of the obtained solutions is real for $b_3c_4 \neq 0$, which finishes this case.

2. $e_2 = 0$: Now we can assume that $e_1 \neq 0$ holds, as otherwise the orientation of the platform is fixed under the 2-dimensional self-motion (\Rightarrow pure translational motion). Under this assumption we can solve $\Psi, \Delta_{3,1}$ for f_1, f_3 w.l.o.g. and plug the obtained solutions into $\Delta_{2,1}, \Delta_{4,1}$ and $\Delta_{5,1}$. The numerators of the resulting expressions are denoted by G_2, G_4 and G_5 , respectively. G_2 equals $N(R_1^2 - R_2^2)$ and G_4, G_5 are homogeneous cubic polynomials in e_0, e_1 .

We eliminate e_0 from G_4 by calculating the resultant of G_4 and N with respect to e_0 which yields:

$$16b_3^2e_1^6(b_4^2 + c_4^2) \left[(b_3 - b_4)^2 + c_4^2 \right].$$

This expression cannot vanish w.c. over \mathbb{R} for $b_3c_4 \neq 0$.

3.2. $\beta = 1$

We attach a sixth leg with anchor points \mathbf{m}_6 and \mathbf{M}_6 in a way that the platform and the base of the resulting hexapod are still similar. This is a so-called equiform hexapod. If the pentapod has a 2-dimensional self-motions of type $\beta = 1$, the equiform hexapod has to have a non-empty set \mathcal{B}_f , which is defined analogously to the one of pentapods (cf. Definition 1 and Eq. (3)).

Due to [17] (and [16] for the special case of congruence) the following necessary condition for the existence of a projected bond $\in \mathcal{B}_f$ of an equiform (congruent) hexapod is known: The anchor points have to be located on a cylinder of revolution over \mathbb{C} .

Therefore the points $\mathbf{m}_1, \dots, \mathbf{m}_6$ of the resulting equiform hexapod have to be located on a cylinder of revolution Φ for any choice of \mathbf{m}_6 . The special choice $\mathbf{m}_6 \in [\mathbf{m}_i, \mathbf{m}_j]$ implies that the line $[\mathbf{m}_i, \mathbf{m}_j]$ has to be contained in Φ . Therefore Φ can only be (cf. [16] or [17]):

- ★ either a cylinder of revolution over \mathbb{R} and $[m_i, m_j]$ is one of its rulings,
- ★ or it splits up into a pair of isotropic planes, which are not conjugate complex. In this case Φ carries two real lines.

Therefore the following statement has to hold:

Given are 5 real points m_1, \dots, m_5 , which are not coplanar. For each of the 10 lines $[m_i, m_j]$ $i \neq j \in \{1, 2, 3, 4, 5\}$ one of the following two properties has to hold:

1. m_1, \dots, m_5 are located on a real cylinder of revolution with the ruling $[m_i, m_j]$.
2. There exists a second line skew to $[m_i, m_j]$ in a way that m_1, \dots, m_5 are located on both lines.

In the following we show that the second possibility yields a contradiction: If m_1, \dots, m_5 are located on two skew lines, then we can assume that no 4 points are collinear, as otherwise we contradict our non-planarity assumption. W.l.o.g. we can assume that m_1, m_2, m_3 are located on the line g . If we consider for example the line $[m_1, m_4]$, it can easily be seen that no cylinder of revolution Φ with the ruling $[m_1, m_4]$ passing through m_2, m_3, m_5 can exist, as g can intersect Φ only in two points. Therefore we can reduce the problem to the following question:

Given are 5 real points m_1, \dots, m_5 , which are not coplanar and where no three points are collinear. Can a configuration of these 5 points exist in a way that they are located on the 10 real cylinders of revolution with rulings $[m_i, m_j]$ for $i \neq j \in \{1, 2, 3, 4, 5\}$?

The answer is "no", which can be proven as follows: There has to exist a plane ε_{ij} containing $[m_i, m_j]$ with the property that the remaining three points are located in the same half-space with respect to ε_{ij} but do not belong to ε_{ij} . The reason for this is that ε_{ij} can be seen as tangent plane to the cylinder along the ruling $[m_i, m_j]$.

Therefore the segment $\overline{m_i m_j}$ has to be an edge of the convex hull of the point set m_1, \dots, m_5 . As a consequence no 4 points of m_1, \dots, m_5 can be coplanar. Therefore the convex hull has to be a convex polyhedron with 5 vertices, 10 edges and 10 faces. As this contradicts Euler's polyhedron formula $V - E + F = 2$, Theorem 2 is proven. \square

4. Proving Theorem 3

Due to the considerations given in Section 2.2 we only have to discuss the cases $\beta = 0$ and $\beta = 1$.

4.1. $\beta = 1$

Assume that a planar affine pentapod with a 2-dimensional self-motion of type $\beta = 1$ is given. Now we attach a sixth leg in a way that we get a planar affine hexapod, which is not architecturally singular.⁵ Therefore the resulting hexapod has to have a non-empty set \mathcal{B}_f . In Example 5 of [11] it is proven that a planar affine hexapod does not possess a projected bond. Therefore a planar affine pentapod cannot have a 2-dimensional self-motions of type $\beta = 1$.

4.2. $\beta = 0$

This case has to be subdivided with respect to the criterion if the affinity α is a congruence transform or not.

4.2.1. The affinity is no congruence transformation

We assume that a pentapod with a 2-dimensional self-motion \mathcal{S} of type $\beta = 0$ is given. Therefore it can perform in each configuration C of \mathcal{S} a 1-dimensional translational self-motion \mathcal{T} . If we disconnect the platform and the base in the pose C and translate the platform in a way that $M_1 = m_1$ holds, then the following relation has to be fulfilled due to the last paragraph of Section 2.1:

$$rk(\overrightarrow{M_2 m_2}, \dots, \overrightarrow{M_5 m_5}) = 1.$$

Now the following two cases have to be distinguished:

⁵This can always be done if the given pentapod is not architecturally singular.

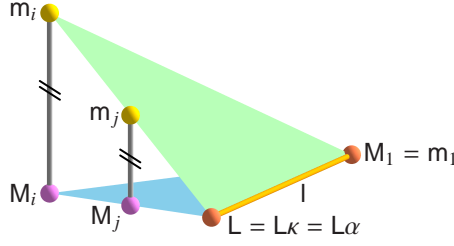


Figure 2: The parallel projection κ from the base plane to the non-coinciding platform plane equals the affinity α .

1. **For all C of \mathcal{S} the platform is parallel to the base:** Therefore \mathcal{S} has to be a Schönflies motion where the direction of the rotational axis is orthogonal to the base plane. In the following we want to compute the affine pentapods with this property:

W.l.o.g. we can assume that M_1 equals in the origin of the fixed system, that M_2 is located on its x -axis and that the remaining base anchor points belong to the xy -plane. The same can be assumed for the platform with respect to the moving system. Moreover we can assume m_1, m_2, m_3 as well as M_1, M_2, M_3 are not collinear (otherwise we can relabel the anchor points). Therefore the (regular) affinity between the platform and base is determined by the first three pairs of anchor points with coordinates:

$$\mathbf{M}_1 = \mathbf{m}_1 = (0, 0, 0)^T, \quad \mathbf{m}_2 = (a_2, 0, 0)^T, \quad \mathbf{m}_3 = (a_3, b_3, 0)^T, \quad \mathbf{M}_2 = (A_2, 0, 0)^T, \quad \mathbf{M}_3 = (A_3, B_3, 0)^T,$$

and $a_2 a_3 b_3 \neq 0$. Now we rotate the platform about the z -axis which yields $\mathbf{m}'_i := \mathbf{R} \mathbf{m}_i$ where e_1 and e_2 have to be set equal to zero in the matrix \mathbf{R} of Eq. (1). Now the necessary condition for the existence of a translational self-motion reads as:

$$(\mathbf{m}'_2 - N\mathbf{M}_2) \times (\mathbf{m}'_3 - N\mathbf{M}_3) = \mathbf{0}.$$

This only implies the following equation:

$$(e_0^2 + e_3^2) \left[(B_3 - b_3)(A_2 - a_2)e_0^2 + (B_3 + b_3)(A_2 + a_2)e_3^2 + 2(A_3 a_2 - A_2 a_3)e_0 e_3 \right].$$

The second factor is fulfilled identically if and only if the platform and the base are congruent, which is the excluded case. Therefore there exist at most two orientations which cause a translational self-motion \mathcal{T} , but for a 2-dimensional self-motion \mathcal{S} of type $\beta = 0$ we need at least a 1-dimensional set of such orientations. Hence we have no solution in this case.

2. **There exists a pose C of \mathcal{S} where the platform is not parallel to the base:** After performing the translation of the platform that $M_1 = m_1$ holds there exists a parallel projection κ which maps M_i to m_i for $i = 1, \dots, 5$ (see Fig. 2). Therefore this mapping equals the affine mapping α between the base and the platform.

As a consequence the points L on the line l of intersection of the platform and the base have to be mapped onto each other. It is a well known fact from linear algebra that there exist at most two distinct directions $\mathbf{d}_1, \mathbf{d}_2$, which are mapped without distortion (i.e. $\|\mathbf{d}_i\| = \|\alpha(\mathbf{d}_i)\|$ for $i = 1, 2$), if α is no congruence transformation. Therefore the line l has to be parallel to \mathbf{d}_1 and $\alpha(\mathbf{d}_1)$, which are equally oriented. Moreover the orientations with translational self-motions are obtained by rotating the platform about the line l . Within this 1-parametric set of orientations there are two poses where the platform and the base coincide. It is impossible that in one of these two flat configurations also \mathbf{d}_2 and $\alpha(\mathbf{d}_2)$ are parallel and equally oriented, as otherwise the affinity has to be congruence transformation. Therefore no bifurcation into another 1-parametric set of orientations causing translational self-motions is possible. Therefore \mathcal{S} has to be a Schönflies motion where the direction of the rotational axis equals \mathbf{d}_1 .

With the obtained information the remaining problem can easily be solved by direct computations as follows: W.l.o.g. we can choose the fixed system that M_1 equals its origin and that the x -axis shows in direction \mathbf{d}_1 . Moreover, m_1 is the origin of the moving system and its x -axis corresponds with the direction $\alpha(\mathbf{d}_1)$. Therefore the anchor points have the following coordinates:

$$\mathbf{M}_1 = \mathbf{m}_1 = (0, 0, 0)^T, \quad \mathbf{M}_j = (A_j, B_j, 0)^T, \quad \mathbf{m}_j = (A_j, kB_j, 0)^T,$$

for $j = 2, \dots, 5$. Due to the planarity of the platform we can assume $k \in \mathbb{R}^+$, where the value 0 has to be excluded⁶ as otherwise α is singular. Moreover we know that \mathcal{S} is a Schönflies motion with respect to \mathbf{d}_1 and therefore the Euler parameters e_2 and e_3 have to be equal to zero.

According to Remark 1 a translation of a leg along \mathbf{d}_1 does not influence the Schönflies motion. Therefore we can set $A_j = 0$ for $j = 2, \dots, 5$ w.l.o.g.. As not all base anchor points are collinear we can also assume that $B_2 \neq 0$ holds. Under this assumption we can solve $\Psi, \Delta_{2,1}$ for f_1, f_3 . Then we plug the obtained solutions into $\Delta_{i,1}$ with $i = 3, 4, 5$. The numerators of the resulting expressions are denoted by G_i and they are homogeneous quadratic in e_0, e_1 . Note that G_i does not depend on f_0 and f_2 . These two unknowns only remain⁷ in Λ_1 .

A projected bond exists if $G_3 = 0, G_4 = 0, G_5 = 0$ and $e_0^2 + e_1^2 = 0$ ($\Leftrightarrow N = 0$) have a common root. Hence we compute the resultant of G_i and N with respect to e_1 which yields

$$16B_2^2B_i^2k^2e_0^4(B_2 - B_i)^2.$$

As a consequence the i th leg has to coincide with the first or second leg for $i = 3, 4, 5$. As the collinearity of four anchor points yields a trivial case of an architecturally singular design (cf. item 8 of Theorem 3 given by Karger [8]), we end up with the manipulator given in item 2 of Theorem 3. This manipulator design is sufficient for the existence of a 2-dimensional self-motion \mathcal{S} due to Remark 1.

4.2.2. The affinity is a congruence transformation

In this case we can assume coordinate systems in the platform and the base in a way that the points are located in the xy -plane and that $\mathbf{M}_i = \mathbf{m}_i$ holds for $i = 1, \dots, 5$. As in Section 3.1.2 it can easily be seen that the set of orientations causing translational self-motions is 2-dimensional given by the condition $e_3 = 0$ in the Euler parameters. Moreover due to the result obtained in item 2 of Section 4.2.1 (cf. footnote 6) we can assume that the ratio $e_1 : e_2$ is not constant during the 2-dimensional self-motion \mathcal{S} as otherwise \mathcal{S} has to be a Schönflies motion, which can only result in a special case of the solution given in item 2 of Theorem 3.

Now we check the remaining cases by means of computer algebra. To do this in a clear way, it is split up with respect to criterion whether three collinear anchor points exist or not.

Remark 4. If no three anchor points are collinear then the pentapod is free of so-called butterfly self-motions (cf. [15, 11]). Therefore the splitting with respect to the existence/non-existence of three collinear anchor points is equivalent with the splitting with respect to the existence/non-existence of butterfly self-motions. \diamond

1. **No three anchor points are collinear:** It is well known (cf. Bricard [2], Chasles [18], Duporcq [19]) that one can add a 1-parametric set of legs to a planar projective pentapod without changing the direct kinematics. The anchor points of these legs are also related by the projectivity and they are located on the conic sections uniquely determined by the given five pairs of anchor points (see Fig. 3a). As a congruence α is only a special case of a projectivity, we can use this result for the problem under consideration. We denote the conic section in the base by \mathbf{c} . Hence the corresponding one in the platform is given by $\mathbf{c}\alpha$. Due to our assumption of the non-collinearity of three points, the conic \mathbf{c} has to be regular; i.e. an ellipse, a hyperbola or a parabola.

W.l.o.g. we can choose the coordinate system in the base in a way that the origin coincides with a finite point $\mathbf{M}_1 \in \mathbf{c}$. Then we can still rotate the coordinate system about this point by any angle $\delta \in [0, 2\pi)$. Now we consider the following four lines with respect to the fixed system:

$$y = 0, \quad x = 0, \quad x = y, \quad x = -y.$$

We assume that δ is chosen in a way that none of these four lines equals the tangent in \mathbf{M}_1 with respect to \mathbf{c} or intersects \mathbf{c} in an ideal point. This assumption can be done w.l.o.g. as there can only exist a finite number of such "bad" choices for δ . Therefore we get the following coordinatization:

$$\begin{aligned} \mathbf{M}_1 &= (0, 0, 0)^T, & \mathbf{M}_2 &= (A_2, 0, 0)^T, & \mathbf{M}_3 &= (0, B_3, 0)^T, \\ \mathbf{M}_4 &= (A_4, A_4, 0)^T, & \mathbf{M}_5 &= (A_5, -A_5, 0)^T, \end{aligned}$$

⁶One can also exclude $k = 1$ as it equals the congruent case, but the following calculation also holds for this case.

⁷It can easily be verified that it is impossible that Λ_1 is also independent of f_0 and f_2 .

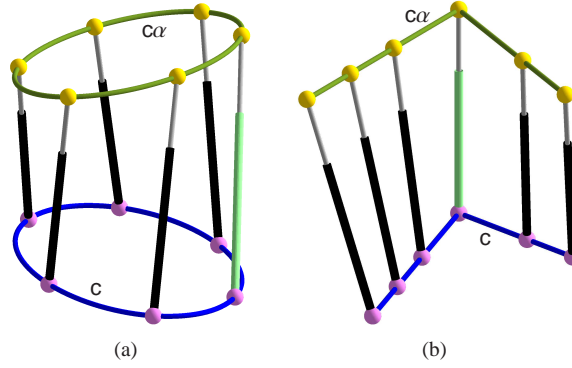


Figure 3: A 1-dimensional set of legs can be attached to a planar projective pentapod. Only one additional leg of this set is visualized in green color: (a) general case (b) conic C splits up into two lines.

with $A_2 B_3 A_4 A_5 \neq 0$. Moreover we can assume w.l.o.g. that $A_2 > 0$ hold; i.e. M_2 is located on the positive x -axis of the fixed frame. The moving frame is chosen analogously thus we get $\mathbf{m}_i = \mathbf{M}_i$ for $i = 1, \dots, 5$.

We solve $\Psi, \Delta_{2,1}$ for f_2, f_3 and plug the obtained solutions into $\Delta_{3,1}, \Delta_{4,1}, \Delta_{5,1}$. The numerators of the resulting expressions are denoted by G_3, G_4 and G_5 , respectively, and they are homogeneous cubic polynomials in e_0, e_1, e_2 . Note that G_3, G_4, G_5 do not depend on f_0 and f_1 . These two unknowns only remain⁸ in Λ_1 .

Now a projected bond exists if the three cubic curves $G_3 = 0, G_4 = 0, G_5 = 0$ have a common intersection point with the conic $e_0^2 + e_1^2 + e_2^2 = 0$ ($\Leftrightarrow N = 0$). In the following we show that this is not possible:

We compute the resultant H_i of G_i and N with respect to e_0 . For H_3 we get $16B_3^2 e_1^2 e_2^2 (B_3 e_1 + e_2)^2$. As e_2 is the denominator of f_2 and f_3 we have to distinguish the following cases:

(a) $e_2 \neq 0$: We remain with two possibilities:

- i. $e_1 = 0$: Then H_j equals $16e_2^6 A_j^2 (A_j - 1)^2$ for $j = 4, 5$. This implies $A_4 = A_5 = 1$, but in this case M_2, M_4, M_5 are collinear.
- ii. $e_1 = -e_2/B_3$: In this case we get

$$H_4 = 16A_4^2 B_3^{-4} e_2^6 (B_3 + 1)^2 (B_3 A_4 - B_3 + A_4)^2, \quad H_5 = 16A_5^2 B_3^{-4} e_2^6 (B_3 - 1)^2 (B_3 A_5 - B_3 + A_5)^2.$$

As the last factor yields the collinearity of M_2, M_3, M_4 and M_2, M_3, M_5 , respectively, we are done.

(b) $e_2 = 0$: Finally we have to show that $(1 : \pm i : 0 : 0)$ is no projected bond of our manipulator. Under consideration of these Euler parameters we can solve $\Psi, \Delta_{3,1}$ for f_1, f_3 which yields $f_1 = \pm i f_0$ and $f_3 = \mp i B_3/2$, respectively. Then $\Delta_{4,1}$ and $\Delta_{5,1}$ can only vanish w.c. for $B_3 = A_4 = -A_5$, but in this case M_3, M_4, M_5 are collinear.

2. **Three anchor points are collinear:** Due to this collinearity the conic section C splits up into two lines (see Fig. 3b). Moreover we can assume that no four anchor points are collinear, as otherwise we get an architecturally singular pentapod.

W.l.o.g. we can assume that M_1, M_2, M_3 are located on the x -axis. Due to the result of Bricard [2], Chasles [18] and Duporcq [19], we can even fix their coordinates as follows:

$$\mathbf{M}_1 = (0, 0, 0)^T, \quad \mathbf{M}_2 = (1, 0, 0)^T, \quad \mathbf{M}_3 = (-1, 0, 0)^T.$$

(a) $[M_4, M_5]$ is not parallel to the x -axis: In this case we can assume w.l.o.g. that their point of intersection is the origin ($= M_1$). Now the remaining two base anchor points can be coordinatized as:

$$\mathbf{M}_4 = (A_4, 1, 0)^T, \quad \mathbf{M}_5 = (-A_4, -1, 0)^T.$$

⁸It can easily be verified that it is impossible that Λ_1 is also independent of f_0 and f_1 .

The coordinates of the corresponding platform anchor points are determined by $\mathbf{m}_i = \mathbf{M}_i$ for $i = 1, \dots, 5$.

We can solve $\Psi, \Delta_{2,3}$ w.l.o.g. for f_2 and f_3 . Then we plug the obtained solutions into $\Delta_{2,1}, \Lambda_4 + \Lambda_5$ and $\Delta_{4,5}$. The numerators of the resulting expressions are denoted by G_2, G_4 and G_5 , respectively, and they do not depend on f_0 and f_1 . G_2 and G_4 are homogeneous quadratic polynomials in e_0, e_1, e_2 and G_5 factors into $(e_0^2 + e_1^2 + e_2^2)H_5$ with:

$$H_5 = (R_3^2 - R_2^2)e_1 + [R_3^2 - R_4^2 - A_4(R_3^2 - R_2^2)]e_2.$$

Therefore H_5 has to be fulfilled identically, as otherwise the ratio $e_1 : e_2$ is constant. This implies $R_2^2 = R_3^2$ and $R_4^2 = R_5^2$.

In order that a 2-dimensional self-motion of type $\beta = 0$ exists, the two remaining equations $G_2 = 0$ and $G_4 = 0$ have to have a common factor. Therefore we compute the resultant of these two expressions with respect to e_0 , which yields $16L^2$ with

$$L = (R_1^2 - R_3^2)e_1^2 + [A_4^2(R_1^2 - R_3^2) - R_1^2 + R_5^2]e_2^2 - 2A_4(R_1^2 - R_3^2)e_1e_2.$$

The necessary condition $R_1^2 = R_3^2$ already implies $G_2 = 4e_2^2$, which cannot vanish w.c..

- (b) $[\mathbf{M}_4, \mathbf{M}_5]$ is parallel to the x -axis: In this case the remaining two platform anchor points can be coordinatized as follows:

$$\mathbf{M}_4 = (0, B_4, 0)^T, \quad \mathbf{M}_5 = (1, B_4, 0)^T.$$

Moreover we can eliminate the factor of scaling by setting $B_4 = 1$. The coordinates of the corresponding platform anchor points are determined by $\mathbf{m}_i = \mathbf{M}_i$ for $i = 1, \dots, 5$.

We can solve $\Psi, \Delta_{2,3}$ w.l.o.g. for f_2 and f_3 . Then we plug the obtained solutions into $\Delta_{2,1}$ and $\Delta_{4,5}$. The numerators of the resulting expressions are denoted by G_2 and G_4 , respectively, and they do not depend on f_0 and f_1 . G_2 and G_4 are homogeneous quadratic polynomials in e_0, e_1, e_2 .

In order that a 2-dimensional self-motion of type $\beta = 0$ exists, the two equations $G_2 = 0$ and $G_4 = 0$ have to have a common factor. Therefore we compute the resultant of these two expressions with respect to e_0 , which yields:

$$64e_2^2 [(2R_1^2 - R_2^2 - R_3^2)e_1 - (R_1^2 - R_2^2 + R_5^2 - R_4^2)e_2]^2.$$

The necessary condition $R_1^2 = (R_2^2 + R_3^2)/2$ already implies $G_2 = 4e_2^2$, which cannot vanish w.c.. This finishes the proof of Theorem 3. \square

5. Proving Theorem 4

As already mentioned in the last paragraph of Section 1 the proof of this theorem splits up into two parts, which are discussed in the following subsections:

5.1. Study of item (c) of Theorem 1

We only have to discuss the cases $p = 2, 3, 4$, as $p < 2$ yields architecturally singular designs and $p = 5$ the excluded case of a linear platform. But within the following study we allow the collinearity of the five base anchor points, as in this way we get the following three additional cases as a side-result:

- (α) We get a special case of item 1 of Theorem 4 if $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5$ are collinear: The 2-dimensional spherical self-motion is obtained if $\mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5$ is located on the line spanned by \mathbf{m}_1 and \mathbf{m}_2 .
- (β) We get a special case of item 2(b) of Theorem 4 if $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4 = \mathbf{M}_5$ are collinear: The 2-dimensional spherical self-motion is obtained if $\mathbf{M}_4 = \mathbf{M}_5$ is located on the carrier line of the collinear points $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$.
- (γ) We get a special case of item 3 of Theorem 4 if $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4, \mathbf{M}_5$ are collinear: The 2-dimensional spherical self-motion is obtained if \mathbf{M}_5 is located on the carrier line of the collinear points $\mathbf{m}_1, \dots, \mathbf{m}_4$.

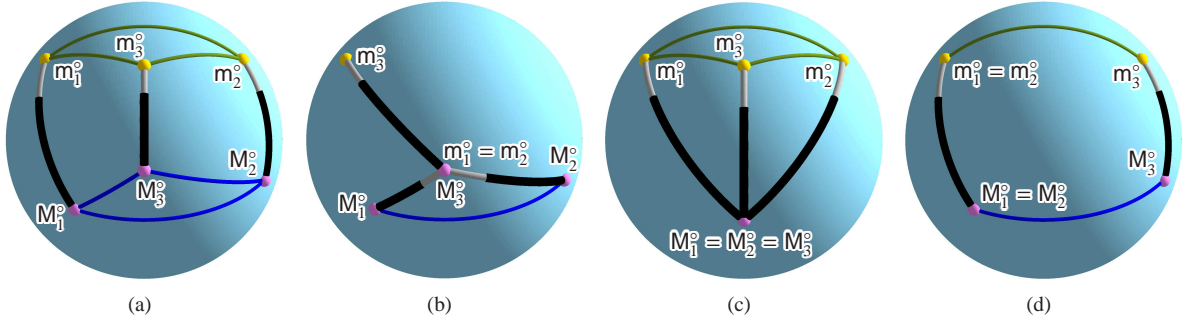


Figure 4: (a) Sketch of a 3-legged spherical 3-dof RPR manipulator. (b) Self-motion of type (I). (c) Self-motion of type (II). (d) 2-legged spherical manipulator (= spherical 4-bar mechanism).

Until now these three special cases were not reported in the literature to the best knowledge of the authors. Hence, their existence necessitates a contemporary and accurate re-examination of the old results on this topic (cf. page 222 of Darboux [20], pages 180ff. of Mannheim [21], Duporcq [22]; see also Chapter III of Bricard [2]), which also takes the coincidence of anchor points into account. This is done by the authors in [6].

For the proof of Theorem 4 (including the three additional cases (α, β, γ)) we need the following result on 3-legged spherical 3-dof RPR manipulators proven in Lemma 2 of [10] and Theorems 5 and 6 of [11]:

A 3-legged spherical 3-dof RPR manipulator with base anchor points $M_1^\circ, M_2^\circ, M_3^\circ$ and platform anchor points $m_1^\circ, m_2^\circ, m_3^\circ$ (see Fig. 4a) can only have a self-motion in the following two cases (under consideration of footnote 1):

- (I) *If $m_2^\circ = m_3^\circ$ coincides with M_1° , then the platform can rotate freely about this point (see Fig. 4b).*
- (II) *If the base degenerates into one point; i.e. $M_1^\circ = M_2^\circ = M_3^\circ$. Then the manipulator can rotate in all poses about this point (see Fig. 4c).*

5.1.1. $p = 2$

We can assume that m_3, m_4, m_5 are not collinear, as otherwise we get an architecturally singular pentapod. Therefore the platform can perform a pure spherical motion about the center $M_3 = M_4 = M_5$. Now the two legs m_4M_4 and m_5M_5 only imply one constraint in one of the following two cases:

1. One anchor point is located in the center: It has to be a platform anchor point as otherwise we get an architecturally singular design. This implies item 1 of Theorem 4.
2. The projection $m_4^\circ M_4^\circ$ and $m_5^\circ M_5^\circ$ of the legs m_4M_4 and m_5M_5 on the unit sphere centered in $M_3 = M_4 = M_5$ coincide. The condition $m_4^\circ = m_5^\circ$ implies that $M_3 = M_4 = M_5$ is located on the line spanned by m_4 and m_5 . The condition $M_4^\circ = M_5^\circ$ implies the collinearity of all 5 base anchor points. Therefore we get the additional case (α) .

For the discussion of the case $p = 3, 4$ we can assume that no three base anchor points coincide, as otherwise we can only get a special case of $p = 2$.

5.1.2. $p = 3$

For an intuitive argumentation of this case, we interchange platform and base; i.e. we have M_1, M_2, M_3 collinear, $m_4 = m_5$ and the assumptions that neither M_1, \dots, M_5 are collinear nor three platform anchor points coincide. As $m_4 = m_5$ can only be located on a circle \mathcal{S} with axis $[M_4, M_5]$, we distinguish the following possibilities:

1. $m_4 = m_5$ is fixed during the 2-dimensional self-motion (\Rightarrow spherical motion with center $m_4 = m_5$): Now the three legs m_iM_i ($i = 1, 2, 3$) are only allowed to imply one constraint. We have to discuss the following three possibilities:
 - (a) If $M_1 = M_2$ is located in the center $m_4 = m_5$. This implies item 2(a) of Theorem 4.

- (b) If M_1 is located in the center and the projection $m_2^o M_2^o$ and $m_3^o M_3^o$ of the remaining two legs on the unit sphere centered in $m_4 = m_5$ coincide; i.e. $m_2^o = m_3^o$ and $M_2^o = M_3^o$. This implies $m_2, m_3, m_4 = m_5$ collinear and we get item 2(b) of Theorem 4.
- (c) The projection $m_i^o M_i^o$ of the three legs on the unit sphere centered in $m_4 = m_5$ coincide. For the same reason as in the last item $m_1, m_2, m_3, m_4 = m_5$ have to be collinear as well as M_1, M_2, M_3 . This yields the additional case (β).
2. $m_4 = m_5$ moves on s : In each pose of $m_4 = m_5$ there has to be a 1-dimensional spherical self-motion with center $m_4 = m_5$. Now it is not possible that one of the M_i 's ($i = 1, 2, 3$) always coincide with $m_4 = m_5$ as this point moves along s . Therefore we can consider the resulting spherical manipulator $m_i^o M_i^o$ for $i = 1, 2, 3$ with respect to the unit sphere centered in $m_4 = m_5$. The resulting spherical manipulator is:
- (a) 3-legged: Due to the cited results (at the beginning of Section 5) a necessary condition for its self-mobility is that either two platform points or two base points coincide:
- i. $M_1^o = M_2^o$: This is only possible for all poses of $m_4 = m_5$ on s if $M_1 = M_2$ holds. Now we have to distinguish between the self-motions (I) and (II):
 - A. $M_1^o = M_2^o$ coincides with m_3^o : We get the 1-dimensional self-motion if $m_3, m_4 = m_5$ and $M_1 = M_2$ are collinear. Due to the fixed leg length the distance from $M_1 = M_2$ to $m_4 = m_5$ has to be constant during the motion of $m_4 = m_5$ on s . Therefore $M_1 = M_2$ has to be located on the axis $[M_4, M_5]$. Moreover as the distance from m_3 to $m_4 = m_5$ is trivially fixed also M_3 has to be located on $[M_4, M_5]$. Therefore all base anchor points have to be collinear; a contradiction.
 - B. $M_1^o = M_2^o = M_3^o$: Therefore $M_1 = M_2 = M_3$ has to hold. As a consequence s has to be a circle on a sphere centered in $M_1 = M_2 = M_3$, which implies the collinearity of $M_1 = M_2 = M_3, M_4, M_5$; a contradiction.
 - ii. $m_1^o = m_2^o$: This implies the collinearity of $m_1, m_2, m_4 = m_5$. Now we have to distinguish between the self-motions (I) and (II):
 - A. $m_1^o = m_2^o$ coincides with M_3^o : We get the 1-dimensional self-motion if $m_1, m_2, m_4 = m_5$ and M_3 are collinear. Due to the fixed leg length the distance from M_3 to $m_4 = m_5$ has to be constant during the motion of $m_4 = m_5$ on s . Therefore M_3 has to be located on the axis $[M_4, M_5]$. Moreover as the distance from m_i ($i = 1, 2$) to $m_4 = m_5$ is trivially fixed also M_i has to be located on $[M_4, M_5]$. Therefore all base anchor points have to be collinear; a contradiction.
 - B. $m_1^o = m_2^o = m_3^o$: Therefore $m_1, m_2, m_3, m_4 = m_5$ are collinear. As no three platform anchor points are allowed to coincide we remain with two possibilities:
 - m_1, m_2, m_3 are pairwise distinct: Therefore the first three legs of the pentapod span a regulus \mathcal{R} of lines. It is well known (e.g. [23]) that each line of \mathcal{R} can be replaced by any other line of \mathcal{R} without changing the singularity-set of the pentapod. E.g. we can replace the first leg by the line of \mathcal{R} through $m_4 = m_5$ (see Fig. 5a). Its base anchor point M cannot be located on the axis $[M_4, M_5]$, as otherwise the originally given pentapod is an architecturally singular one (special case of item 10 of Theorem 3 given in [8]; see also item 9 of Corollary 1 given in [6]). Therefore this already shows that $m_4 = m_5$ is fixed, which contradicts our assumption that the point moves along s .
 - m_1, m_2, m_3 are not pairwise distinct: W.l.o.g. we can assume $m_1 = m_2$. Now M_3 cannot be located on the axis $[M_4, M_5]$, as otherwise the pentapod is architecturally singular (item 7 of Theorem 3 of [8]). Now we can perform the following singular-invariant leg-rearrangement according to [23]: We can replace the leg $m_1 M_1$ by the leg $m_4 M_3$ (see Fig. 5b). This again shows that $m_4 = m_5$ is fixed; a contradiction.
- (b) 2-legged: W.l.o.g. we can assume $M_1^o = M_2^o$ and $m_1^o = m_2^o$ (see Fig. 4d). This implies $M_1 = M_2$ and the collinearity of $m_1, m_2, m_4 = m_5$. $M_1 = M_2$ cannot be located on the axis $[M_4, M_5]$, as otherwise we get an architecturally singular pentapod (special case of item 8 of Theorem 3 given in [8]). Now we can perform the following singular-invariant leg-rearrangement according to [23]: We can replace the leg $m_1 M_1$ by the leg $m_4 M_1$ (see Fig. 5c). This already shows that $m_4 = m_5$ is fixed; a contradiction.

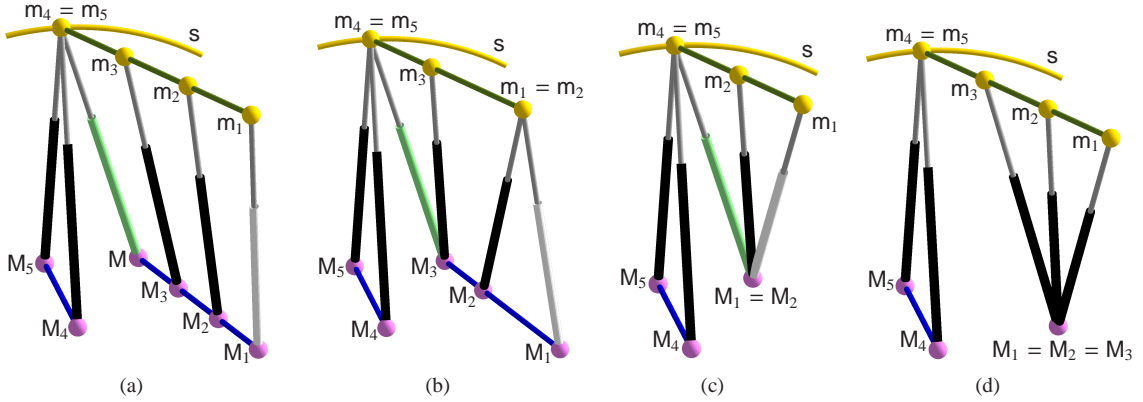


Figure 5: (a,b) Sketches of the two case listed in item 2(a)iiB of $p = 3$. (c,d) Sketch of case 2(b) and 2(c), respectively, of $p = 3$.

- (c) 1-legged: From $m_1^\circ = m_2^\circ = m_3^\circ$ we get the collinearity of $m_1, m_2, m_3, m_4 = m_5$. Moreover $M_1^\circ = M_2^\circ = M_3^\circ$ implies $M_1 = M_2 = M_3$ and therefore the first three legs are in a pencil of lines (see Fig. 5d). This is an architecturally singular case (item 6 of Theorem 3 given in [8]).

5.1.3. $p = 4$

For the discussion of the case m_1, \dots, m_4 collinear we can assume that M_5 does not coincide with another base anchor point, as otherwise we can only get a special case of $p = 3$. If we remove the fifth leg, then we get a quadropod, which can have a 2-dimensional self-motion or a higher one:

- 2-dimensional self-motion: Now this motion consists of a 1-dimensional rotation of the platform about the carrier line g of m_1, \dots, m_4 and a 1-dimensional motion of g itself. As we can assume that m_5 is not located on g , the point M_5 has to be located on g in each pose of the self-motion. Moreover as the leg lengths is fixed M_5 has to coincide with the same point $G \in g$.

Now it remains to guarantee the 1-dimensional motion of g , which can only be a spherical one due to $M_5 = G$ (center of the spherical motion).

- If $G \neq m_i$ holds for all $i \in \{1, 2, 3, 4\}$, then there can only be a motion of g if M_1, \dots, M_5 are collinear. This yields the additional case (γ).
 - If $G = m_1$ holds, then only M_2, \dots, M_5 have to be collinear. This yields item 3 of Theorem 4.
 - If $G = m_1 = m_2$ holds, then only M_3, \dots, M_5 have to be collinear. This yields again item 2(b) of Theorem 4.
 - If $G = m_1 = m_2 = m_3$ holds. This yields again item 1 of Theorem 4.
- higher-dimensional self-motion: The quadropod can only have an n -dimensional self-motion with $n > 2$ in one of the following cases (according to [5]) if it is not architecturally singular:
 - If $m_1 = m_2 = m_3$ coincides with M_4 : This case implies once more item 1 of Theorem 4.
 - If $m_1 = m_2$ coincides with $M_3 = M_4$: This case implies again item 2(a) of Theorem 4 (with the extra condition m_1, \dots, m_4 collinear). This finishes the discussion of item (c) of Theorem 1.

5.2. Study of item (d) of Theorem 1

Due to the discussion done in Section 5.2, we can assume the following for the pentapod design given in item (d) of Theorem 1: $M_4 \neq M_5$, $m_4 \neq m_5$, and neither $M_1 = M_2 = M_3$ nor $m_1 = m_2 = m_3$ holds.

Moreover the first, second and third leg span a regulus \mathcal{R} (which can degenerate into two pencils of lines). W.l.o.g. we can assume that M_1, M_2, M_3 are pairwise distinct, as otherwise we can replace one of the three legs by a line of \mathcal{R}

without changing the singularity-set of the pentapod (cf. [23]). Due to this possibility of leg-replacements within \mathcal{R} we have even a free choice of pairwise distinct anchor points M_1, M_2, M_3 on the line \mathfrak{g} .

Moreover the distance between \mathfrak{g} and \mathfrak{h} can be assumed to be 1, which eliminates the scaling factor. Therefore the base anchor points of the pentapod can be coordinatized as follows:

$$\begin{aligned} \mathbf{M}_1 &= (0, 0, 0)^T, & \mathbf{M}_2 &= (0, 1, 0)^T, & \mathbf{M}_3 &= (0, -1, 0)^T, \\ \mathbf{M}_4 &= (1, 0, 0)^T, & \mathbf{M}_5 &= (1, B, 0)^T. \end{aligned}$$

The platform has the following coordinatization:

$$\begin{aligned} \mathbf{m}_1 &= (0, 0, 0)^T, & \mathbf{m}_2 &= (0, b_2, 0)^T, & \mathbf{m}_3 &= (0, b_3, 0)^T, \\ \mathbf{m}_4 &= (a, b_4, 0)^T, & \mathbf{m}_5 &= (a, b_5, 0)^T, \end{aligned}$$

where $a > 0$ can be assumed w.l.o.g.. Note that $aB(b_4 - b_5) \neq 0$ holds and that $b_2 = b_3 = 0$ yields a contradiction.

The 2-dimensional self-motion \mathcal{S} of this pentapod can be assumed to be of type $\beta = 1$, as the existence of a translational self-motions implies an affine coupling of the planar platform and the planar base (see Fig. 2 under consideration of the last paragraph of Section 2.1), which was already discussed in Section 4.

The triangles M_1, M_4, M_5 and m_1, m_4, m_5 determine uniquely a regular affinity $\mu: M_i \mapsto m_i$ for $i = 1, 4, 5$. As not $\mu(M_2) = m_2$ and $\mu(M_3) = m_3$ can hold simultaneously (otherwise we get again an planar affine pentapod) we can assume w.l.o.g. that $\mu(M_2) \neq m_2$ holds, which equals the condition:

$$Bb_2 + b_4 - b_5 \neq 0. \quad (4)$$

Based on this preparatory work/assumptions we prove by direct computations that the set of projected bonds \mathcal{B}_f cannot be 1-dimensional, which shows that item (d) of Theorem 1 contains no further solutions beside the affine ones listed in Theorem 3. This more technical part of the proof is given in the Appendix in order to streamline the presentation and to improve the readability of the paper. \square

6. Conclusions for hexapods

The adding of an arbitrary leg to any pentapod with mobility 2 yields in the generic case a hexapod with mobility 1. Now the question arises for those cases where the attachment does not restrict the dimension of the mobility. Moreover we are only interested in non-architecturally singular hexapods as they have a practical application in robotics as so-called Stewart-Gough platforms. In the following we give a complete list of these manipulators:

Theorem 5. *A non-architecturally singular hexapod with mobility 2 belongs to one of the following six cases (under consideration of footnote 1):*

1. *Platform and base are congruent: We get the trivial 2-dimensional translation of the congruent hexapod.*
2. *m_1, \dots, m_5 collinear and M_2, \dots, M_6 collinear: We get a 2-dimensional spherical self-motion if m_1 coincides with M_6 .*
3. *$m_1 = m_2, m_3, m_4, m_5$ collinear and M_3, \dots, M_6 collinear: We get a 2-dimensional spherical self-motion if $m_1 = m_2$ coincides with M_6 .*
4. *$m_1 = m_2 = m_3, m_4, m_5$ collinear and M_4, M_5, M_6 collinear: We get a 2-dimensional spherical self-motion if $m_1 = m_2 = m_3$ coincides with M_6 .*
5. *$m_1 = m_2, m_3, m_4$ collinear and $M_3, M_4, M_5 = M_6$ collinear: We get a 2-dimensional spherical self-motion if $m_1 = m_2$ coincides with $M_5 = M_6$.*
6. *$m_1 = m_2 = m_3$ and $M_5 = M_6$: We get a 2-dimensional spherical self-motion if $m_1 = m_2 = m_3$ coincides with $M_5 = M_6$.*

PROOF: Clearly architecturally singular pentapods yield architecturally singular hexapods by the attachment of an arbitrary leg. Therefore we can focus on non-architecturally singular pentapods with mobility 2. They are given in the Theorems 2, 3 and 4 or belong to the case $p = 5$ of item (c) of Theorem 1.

As the 2-dimensional translation of the congruent hexapod is trivial we can proceed with item 2 of Theorem 3. We assume that our hexapod possesses the 2-dimensional Schönflies self-motion described in item 2 of Theorem 3. If we take any of the six legs away we have to end up with the pentapod of item 2 of Theorem 3 up to permutation of indices. As a consequence the six anchor points have to be located on two parallel lines, where each line carries three anchor points. Therefore the anchor points are located on a degenerated conic, which already shows that the hexapod is architecturally singular.

In the following we investigate the cases listed in Theorem 4. We start with item 1; i.e. $M_3 = M_4 = M_5$: We consider the 2-dimensional self-motion of this pentapod when $M_3 = M_4 = M_5$ coincides with m_1 . This spherical self-motion is not restricted by an additional leg m_6M_6 in one of the following cases:

- ★ $M_6 = M_3 = M_4 = M_5$: This yields an architecturally singular hexapod.
- ★ $m_6 = m_1$: This yields item 6 of Theorem 5.
- ★ $M_3 = M_4 = M_5, M_2, M_6$ collinear and m_1, m_2, m_6 collinear: This yields item 4 of Theorem 5.

The discussion for the items 2(a), 2(b) and 3 of Theorem 4 can be done analogously. They already imply all the remaining cases given in Theorem 5.

Therefore we are left with the pentapods belonging to the case $p = 5$ of item (c) of Theorem 1. Due to Corollary 1 and the list of architecturally singular pentapods (cf. Theorem 3 of [8] under consideration of [9]) there does not exist a pentapod of this type with mobility 3 (or higher), which is not architecturally singular. Therefore all non-architecturally singular pentapods of this type have mobility 2, where one degree of freedom is the rotation about the carrier line g of m_1, \dots, m_5 . This mobility is not restricted by the attachment of a sixth leg only in the already obtained five cases 2–6 of Theorem 5. This can easily be seen by performing analogous considerations as in the proof of Theorem 4, case $p = 4$, item 1. \square

Remark 5. Note that the items 1 and 6 of Theorem 5 were already listed in [5]. The remaining four cases are new to the best knowledge of the authors. Moreover this listing also shows that there does not exist a non-architecturally singular hexapod with a 2-dimensional self-motion of type $\beta = 0$, which was also an open question of [5]. \diamond

Due to Theorem 5 and the results of [5] only the classification of all non-architecturally singular hexapods with a 1-dimensional self-motion remains for the complete solution of the famous Borel-Bricard problem. Some necessary conditions for these overconstrained mechanisms were already presented in [15]. Further investigations on this topic are dedicated to future research.

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Appendix

This technical part of the proof started in Section 5.2 splits up into a general case and a special one. We recommend to study them together with the corresponding Maple worksheets, which can be downloaded from the homepage (<http://www.geometrie.tuwien.ac.at/nawratil>) of the first author.

General case: $e_0 e_1(a - 1) - e_2 e_3(a + 1) \neq 0$

Due to this assumption and Eq. (4) we can solve $\Psi, \Delta_{2,1}, \Delta_{4,1}, \Delta_{5,1}$ for f_0, \dots, f_3 w.l.o.g.. Then we plug the obtained solutions into the remaining two equations Λ_1 and $\Delta_{3,1}$. The numerators of the resulting expressions are denoted by $G_1[66139]$ and $G_3[160]$, where the number in the brackets gives the number of terms. Note that $G_1 = 0$ represents a surfaces of degree 8 in the Euler parameter space P^3 , in contrast to $G_3 = 0$ and $N = 0$ which are quadrics.

Then we eliminate e_0 by computing the resultant H_j of G_j and N with respect to e_0 for $j = 1, 3$. We get $H_3[170]$ and H_1 is a perfect square; i.e. $H_1 = K[380]^2$. Moreover in H_3 and K only even powers of e_3 appear and therefore we can substitute e_3^2 by \bar{e}_3 which yields \bar{H}_3 and \bar{K} , respectively. Finally we compute the resultant of \bar{H}_3 (quadratic in \bar{e}_3) and \bar{K} (quartic in \bar{e}_3) with respect to \bar{e}_3 which yields $L[167161]$.

A necessary condition for the existence of a common curve of the octic surface and the two quadrics ($\Leftrightarrow \mathcal{B}_f$ is 1-dimensional $\Leftrightarrow \beta = 1$) is that L is fulfilled identically for all e_1, e_2 . As L factors into

$$L = e_1^4 e_2^4 a^2 B(b_4 - b_5)(Bb_2 + b_4 - b_5)^2 U[497]V[5030],$$

where $V = 0$ is pseudo-solution implied by the elimination process⁹, we remain with $U = 0$, which is a homogeneous quartic equation in e_1, e_2 . We denote the coefficient of $e_1^i e_2^j$ of U by $U_{i,j}$ and end up with the following three conditions:

$$U_{4,0} = U_{0,4} = (b_2 + b_3)^2 W_1[44] = 0, \quad U_{3,1} = -U_{1,3} = (b_2 + b_3)W_2[81] = 0, \quad U_{2,2} = W_3[125] = 0.$$

For $b_2 = -b_3$ the condition $W_3 = 0$ simplifies to $b_2^2(Bb_2 + b_4 - b_5)^2 = 0$, which cannot vanish w.c.. Therefore we can assume $b_2 \neq -b_3$, which implies $W_1 = W_2 = W_3 = 0$. Now we eliminate b_5 by calculating the resultant E_k of W_i

⁹This can easily be checked by computing the resultant $G_{1,3}$ of G_1 and G_3 with respect to e_0 , which is of degree 8 in \bar{e}_3 . Then the resultant of $G_{1,3}$ and H_3 (or K) with respect to \bar{e}_3 does not contain the factor V .

and W_j with respect to b_5 for pairwise distinct $i, j, k \in \{1, 2, 3\}$. Then the greatest common divisor GCD of E_1, E_2, E_3 equals $aB^4(b_2 + b_3)^2b_2b_3(b_2 - b_3)T$ with

$$T = (a^2 + b_4^2)(b_2^2 + b_3^2) - 2ab_2b_3(b_2 - b_3) - 2b_2b_3b_4(b_2 + b_3) + 2b_2^2b_3^2.$$

In the following case study we show that the factors of GCD does not imply a solution to our problem:

1. $b_2 = 0$: Now W_1 can only vanish w.c. for:

(a) $b_5 = -a$: Then $W_2 = 0$ implies $b_4 = -Ba - a$. Finally W_3 equals $B^2a^3b_3^4(B^2 + 2B + 2)$ which cannot vanish w.c. over \mathbb{R} .

(b) $b_4 = -Ba - a$ and $b_5 \neq -a$: Now $W_2 = 0$ already yields the contradiction.

2. Analogously the cases $b_3 = 0$ and $b_2 = b_3$ can be studied, as they are geometrically identical to the case $b_2 = 0$ (\mathcal{R} splits up into two pencils of lines). As they also yield no solution, we can assume $b_2b_3(b_2 - b_3) \neq 0$ for the discussion of the remaining case $T = 0$.

3. $T = 0$: We can solve $T = 0$ w.l.o.g. for a which yields:

$$a = \frac{b_2b_3(b_2 - b_3) \pm i \left[b_2^2(b_3 - b_4) + b_3^2(b_2 - b_4) \right]}{b_2^2 + b_3^2}.$$

As a has to be a real number the expression in the brackets has to vanish, which implies $b_4 = \frac{b_2b_3(b_2+b_3)}{b_2^2+b_3^2}$. Then the common factors of W_1, W_2, W_3 are $b_2b_3(b_2 - b_3)J[45]$. J is quadratic in b_5 and can be solved for this unknown w.l.o.g.. Now b_5 can only be real if $B = 2\frac{b_2-b_3}{b_2+b_3}$ holds which yields

$$b_5 = \frac{(3b_2^2 - 2b_2b_3 + 3b_3^2)b_2b_3}{(b_2 + b_3)(b_2^2 + b_3^2)}.$$

In this case \overline{H}_3 only depends on e_1 and e_2 (and not longer on \overline{e}_3) and reads as $e_1^2e_2^2b_2^2b_3^2(b_2 + b_3)^{-2}(b_2 - b_3)^4$.

Therefore $e_1 = 0$ or $e_2 = 0$ has to hold. In both cases $\overline{K} = 0$ implies $b_2^2 - b_2b_3 + 2b_3^2 = 0$, which has no real solution.

Due to this discussion we can assume $GCD \neq 0$ and therefore we can delete all factors of E_1, E_2, E_3 , which are also contained within GCD . We denote the remaining expressions by $E_1^*[464]$, $E_2^*[271]$ and $E_3^*[95]$, where E_3^* can be factored but this is not of importance for that what follows. We compute the resultant D_k of E_i^* and E_j^* with respect to B for pairwise distinct $i, j, k \in \{1, 2, 3\}$. As all common factors of D_1, D_2, D_3 imply $GCD = 0$, we get a contradiction.

Special case: $e_0e_1(a - 1) - e_2e_3(a + 1) = 0$

1. $e_2 \neq 0$: Under this assumption we can solve $e_0e_1(a - 1) - e_2e_3(a + 1) = 0$ for e_3 w.l.o.g..

(a) μ is no orientation preserving congruence transformation: Under this assumption we can solve $\Psi, \Delta_{4,1}, \Delta_{5,1}$ for f_1, f_2, f_3 w.l.o.g.. Then we plug the obtained solutions into $\Delta_{2,1}$ and $\Delta_{3,1}$. The numerators of the resulting expressions are denoted by $G_2[865]$ and $G_3[865]$, which are both of degree 6 in e_0, e_1, e_2 . Then we eliminate e_0 by computing the resultant H_j of G_j and N with respect to e_0 for $j = 2, 3$. These expressions factor into $H_i = a^2e_1^2e_2^2K_i[54]^2$. Therefore both expressions K_2 and K_3 have to be fulfilled independently of e_1, e_2 . It can easily be seen that the resulting system of equations has no solution without yielding a contradiction.

(b) μ is an orientation preserving congruence transformation; i.e. $a = 1, b_4 = 0$ and $b_5 = B$: Then $\Psi, \Delta_{2,1}, \Delta_{5,1}$ can be solved for f_1, f_2, f_3 w.l.o.g.. We plug the obtained solutions into $\Delta_{4,1}$ and denote the numerator by $G_4[14]$, which is again of degree 6 in e_0, e_1, e_2 . Then we eliminate e_0 by computing the resultant H_4 of G_4 and N with respect to e_0 which yield $B^2e_1^2e_2^2(Be_1 - e_2)^2$. This expression cannot vanish w.c..

2. $e_2 = 0$: This implies $a = 1$. We distinguish the following two subcases:

(a) μ is no orientation reversing congruence transformation: The discussion can be done similarly to item 1(a).

(b) μ is an orientation reversing congruence transformation; i.e. $b_4 = 0$ and $b_5 = -B$: The discussion can be done similarly to item 1(b). This closes the proof of Theorem 4. \square